The supersonic motion of an aerofoil through a temperature front

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SUMMARY

A solution is given for the sound waves generated when an infinite wedge passes at supersonic velocity through the plane interface between different media. The results are then applied to the particular case when the two media consist of the same perfect gas at different temperatures. It is observed that the flow can be deduced for any symmetrical aerofoil of infinite span by superposition, and the case of a double wedge is considered in detail.

INTRODUCTION

When an aerofoil passes through the surface separating two media, as for example into a cloud, or through a temperature front, certain transient effects on the lift and drag are to be expected during the transition from one regime of steady flow to another. The transient effects will be examined by the method used in an earlier paper (Craggs 1956) on the reflection of sound waves.

The method used is first to examine the motion due to an infinite wedge moving with constant velocity. The fluid motion may then be expected to possess dynamic similarity, the velocity distribution being the same at all times after the wedge has broken the surface, except for a scale factor depending on the depth of penetration. The solution for the wedge can then be extended to any symmetric aerofoil by the principle of superposition, within the limitations imposed by the linearized theory. It is assumed throughout that the fluid motion is sufficiently small for linearized theory to be adequate, and that the motion is strictly two-dimensional.

1. THE EQUATION OF SOUND WAVES WITH DYNAMIC SIMILARITY

Use polar coordinates (r, θ) based on the point where the apex of the wedge cuts the surface of separation of the two media. Let t be the time measured from the instant at which the apex breaks the surface. Let c be the local velocity of sound and s the condensation, measured by the proportional increase in density due to fluid motion. Then the fluid velocity $\mathbf{q}$ is given by

$$\frac{\partial \mathbf{q}}{\partial t} = -c^2 \nabla s, \tag{1}$$

and the equation of motion is

$$c^2\nabla^2 s = \partial^2 s/\partial t^2. \tag{2}$$

Make the assumption of dynamic similarity and use λ, θ as independent variables, then (2) becomes

$$\lambda^2(1-\lambda^2/c^2)\partial^2 s/\partial\lambda^2+\lambda(1-2\lambda^2/c^2)\partial s/\partial\lambda+\partial^2 s/\partial\theta^2 = 0. \tag{3}$$

Equation (3) is the same as the equation arising in the linearized theory of steady three-dimensional flow when the cone-field method is used (Ward 1955, p. 136) and the treatment here is similar.

For $\lambda > c$ equation (3) is hyperbolic and can be reduced to the canonical form

$$\frac{\partial^2 s}{\partial\mu^2}-\frac{\partial^2 s}{\partial\theta^2} = 0 \tag{4}$$

by the substitution

$$\sec\mu = \lambda/c. \tag{5}$$

The general solution is then of the form

$$s = f(\mu-\theta)+g(\mu+\theta) \tag{6}$$

where f and g are arbitrary functions, and particular cases can be solved by the method of characteristics.

Equation (3) is elliptic for $\lambda < c$ and is reduced by the substitution

$$\lambda/c = \operatorname{sech}(-\nu) \tag{7}$$

to the form

$$\frac{\partial^2 s}{\partial\nu^2}+\frac{\partial^2 s}{\partial\theta^2} = 0. \tag{8}$$

Write s as the real part of a complex function w, where

$$w = s+i\tau, \tag{9}$$

then from (8)

$$w = w(\nu+i\theta) \tag{10}$$

and

$$\frac{\partial s}{\partial\nu} = \frac{\partial\tau}{\partial\theta}, \qquad \frac{\partial s}{\partial\theta} = -\frac{\partial\tau}{\partial\nu}.$$

2. The solution in the regions where the equation is hyperbolic

For simplicity, assume that the wedge moves in the direction normal to the surface separating the two media. Let the motion be in the direction $\theta = \frac{1}{2}\pi$, and let ρ_1, ρ_2 be the densities in the regions $0 > \theta > -\pi$ and $0 < \theta < \pi$, respectively. Use the suffixes 1 and 2 consistently to refer to the two media. Consider a plane with polar coordinates λ, θ (figure 1). Then in the regions $0 < \theta < -\pi$, $\lambda > c_2$ and $0 > \theta > -\pi$, $\lambda > c_1$ the equations are of hyperbolic type, the characteristics being the tangents to the sonic circles, $\lambda = c_2$, $\lambda = c_1$, respectively.

Consider first the motion of the wedge with velocity $V > c_1$ before the apex reaches the point $r = 0$. Then the motion is the familiar steady motion of a wedge with supersonic velocity and the solution is well known. There is a weak shock wave at the Mach angle $\phi_1 = \sin^{-1}(c_1/V)$ to the direction of motion, attached at the apex of the wedge. In front of the shock wave the velocity of the fluid is zero and the condensation is also zero. Behind the shock wave the condensation is

$$s_1 = \sigma_1 = M_1 \delta \sec\phi_1, \tag{11}$$

where M_1 is the Mach number, $M_1 = V/c_1$, and δ is the semi-angle of the wedge.

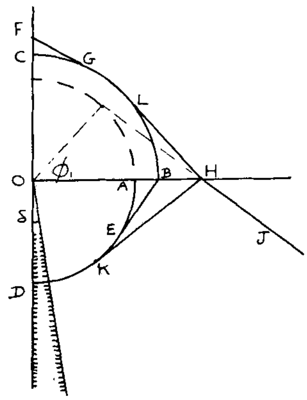

Figure 1.

To discuss the motion after the wedge has broken the surface, measure t from the instant at which the apex reaches the surface. Then the condition of steady motion gives the motion as $t \to 0+$, $\lambda \to \infty$. The boundary conditions on the equations as $\lambda \to \infty$ are therefore

$$\left.\begin{array}{lll} & s_1 = \sigma_1, & -\tfrac{1}{2}\pi < \theta < -\tfrac{1}{2}\pi + \phi_1, \\ & s_1 = 0, & -\tfrac{1}{2}\pi + \phi_1 < \theta < 0, \\ \text{and} & s_2 = 0, & 0 < \theta < \tfrac{1}{2}\pi, \end{array}\right\} \tag{12}$$

with similar conditions for $\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}(3\pi)$.

Use of characteristics theory shows that there is a weak shock wave, of trace

$$\lambda = c_1 \sec(\phi_1 - \theta) \tag{13}$$

and of strength σ_1. This is represented by the line HJ in figure 1.

It is convenient to assume that $c_1 \sec\phi_1 > c_2$, so that the point H, where the shock wave cuts $\theta = 0$ lies outside of the circle $\lambda = c_2$. (Only slight alteration of the argument is needed when this assumption is false, and for the application contemplated, in which the difference in properties between the two media is small, the assumption certainly holds.) Further simple deductions from the theory of characteristics are that there are weak shock waves HK, HL corresponding to the reflection and refraction of the incident wave HJ at the interface, and a new shock wave FG attached at the apex of the wedge. The condensation behind FG is

$$s_2 = \sigma_2 = M_2 \delta \sec \phi_1, \tag{14}$$

where $\phi_2 = \sin^{-1}(c_2/V)$. The conditions behind HK, HL can be deduced from the shock wave equations and the continuity of pressure

$$\rho_1 c_1^2 s_1 = \rho_2 c_2^2 s_2, \tag{15}$$

and of normal velocity $$c_1^2 \partial s_1/\partial\theta = c_2^2 \partial s_2/\partial\theta, \tag{16}$$

over the interface $\theta = 0$. The values are

$$s_1 = 2\sigma_1 \tan\phi_1/\{\tan\phi_1 + k(m^2\sec^2\phi_1 - 1)^{1/2}\} = \sigma_1' \tag{17}$$

and $$\sigma_2' = m^2 k \sigma_1', \tag{18}$$

respectively, where $m = c_1/c_2$, $k = \rho_1/\rho_2$.

The condensations are now known throughout the regions $\lambda > c_1$, $\lambda > c_2$ except for the triangular region ABE, and in this region the theory of characteristics shows only that

$$s_1 = \sigma_1 + f_1(\theta - \mu), \tag{19}$$

the function g of equation (6) being here constant and equal to σ_1.

3. Regions where the equations are elliptic

To discuss the region $0 < \theta < \frac{1}{2}\pi$, $\lambda < c_2$ in which the equation for the condensation is elliptic, it is convenient to introduce a transformation

$$\zeta_2 = \xi_2 + i\eta_2 = \operatorname{sech} 2(\nu_2 + i\theta) \tag{20}$$

which maps the region on the upper half of the ζ plane. The closed boundary of the elliptic region maps into the axis $\eta_2 = 0$, and the points O, A, B, L, G, C of figure 1 correspond to the points

$$\zeta_2 = 0, m^2/(2-m^2), 1, m^2\sec^2\phi_1/(2-m^2\sec^2\phi), \sec^2\phi_2(2-\sec^2\phi_2), -1, \tag{21}$$

respectively.

Now consider the section AB of the interface. On the lower side of AB s_1 is governed by a hyperbolic equation, and from (19)

$$\frac{\partial s_1}{\partial\theta} = \frac{\partial s_1}{\partial\mu} = \lambda\left(\frac{\lambda^2}{c_1^2} - 1\right)^{1/2} \frac{\partial s_1}{\partial\lambda},$$

using (5). The continuity conditions (15) and (16) therefore give

$$\lambda(1-\lambda^2/c_1)^{1/2}\frac{\partial s_2}{\partial\lambda} = k\frac{\partial s_2}{\partial\theta} = -k\frac{\partial\tau_2}{\partial\nu} = -k\lambda\left(1-\frac{\lambda^2}{c^2}\right)^{1/2}\frac{\partial\tau_2}{\partial\lambda}, \tag{22}$$

using (7). This relation holds on the line AB, and on this line

$$\zeta_2 = \xi_2 = \lambda^2/(2c_2^2 - \lambda^2),$$

so $$\frac{dw_2}{d\zeta_2} = \left\{1 - \frac{i}{m^2k}\left(\frac{(2-m^2)\zeta_2 - 1}{1-\zeta_2}\right)^{1/2}\right\}F(\zeta_2) \tag{23}$$

on the line, where $F(\zeta_2)$ is real on the line, and by the principle of analytic continuation equation (23) holds throughout the region $\eta_2 > 0$.

The function $F(\zeta_2)$ is to be determined subject to the following conditions.

(i) Singularities of F are to be expected only at the points (21) and at at any zero of the function in brackets in (23).

(ii) For finite pressures, no singularity of $dw_2/d\zeta_2$ may be of higher order than a simple pole.

(iii) The point at infinity in the ζ_2 plane is an ordinary point, since it corresponds to an ordinary point, $\theta = \frac{1}{4}\pi$, $\lambda = c_2$ in the (λ, θ) plane.

(iv) From the conditions on the quarter circle BGC,

$$\frac{ds_2}{d\xi_2} = 0$$

for $\eta_2 = 0$, $-\infty < \xi_2 < -1$, $1 < \xi_2 < \infty$.

(v) The normal velocity on the surface of the wedge is constant, so $\partial\tau_2/\partial\theta = \partial s_2/\partial\lambda = 0$. On the corresponding section of the real axis $\eta_2 = 0$, $-1 < \xi_2 < 0$, the condition $\partial s_2/\partial\zeta_2 = 0$ holds.

(vi) A careful examination of the conditions in the immediate neighbourhood of the point A of figure 1, with use of the continuity conditions across the interface, shows that $F(\zeta_2)$ can contain only half-integer powers of the quantity $(2-m^2)\zeta_2 - m^2$.

A sufficiently general form for w_2 is then given by

$$\pi\frac{dw_2}{d\zeta_2} = \left[\frac{mk(1-\zeta_2)^{1/2} + i\{(2-m^2)\zeta_2 - m^2\}^{1/2}}{m^2k^2(1-\zeta_2) + (2-m^2)\zeta_2 - m^2}\right] \times \\ \times \left[\frac{P_2 + Q_2\zeta_2^{-1/2}\{(2-m^2)\zeta_2 - m^2\}^{1/2}}{mk(1+\zeta_2)^{1/2}}\right], \quad (24)$$

with
$$P_2 = \frac{B}{(2-m^2\sec^2\phi_1)\zeta_2 - m^2\sec^2\phi_1} + \frac{C}{(2-\sec^2\phi_2)\zeta_2 - \sec^2\phi_2} \quad (25)$$

and
$$Q_2 = \frac{D}{(2-m^2\sec^2\phi_1)\zeta_2 - m^2\sec^2\phi_1} + \frac{E}{(2-\sec^2\phi_2)\zeta_2 - \sec^2\phi_2}. \quad (26)$$

For the solution in the region $\lambda < c_1$, $-\frac{1}{2}\pi < \theta < 0$, a similar transformation

$$\zeta_1 = \operatorname{sech} 2(\nu_1 + i\theta)$$

is used, and (24), together with the continuity conditions across the interface OA, leads to

$$m^2k\pi\frac{dw_1}{d\zeta_1} = \left[\frac{k\{1-(2m^2-1)\zeta_1\}^{1/2} - (1-\zeta_1)^{1/2}}{k^2\{1-(2m^2-1)\zeta_1\} - (1-\zeta_1)}\right] \times \\ \times \left[\frac{P_1 + ikQ_1\zeta_1^{-1/2}\{1+(2m^2-1)\zeta_1\}^{1/2}}{m^2k(1+\zeta_1)^{1/2}}\right], \quad (27)$$

with
$$P_1 = \frac{B}{(2-\sec^2\phi_1)\zeta_1 - \sec^2\phi_1} + \frac{m^2C}{(2m^2-\sec^2\phi_2)\zeta_1 - \sec^2\phi_2} \quad (28)$$

and
$$Q_1 = \frac{D}{(2-\sec^2\phi_1)\zeta_1 - \sec^2\phi_1} + \frac{m^2E}{(2m^2-\sec^2\phi_2)\zeta_1 - \sec^2\phi_2}. \quad (29)$$

The constants B, C, D, E are easily determined by an examination of the singularities at the points corresponding to G, L, K of figure 1. At G, for example, where $\zeta_2 = \sec^2\phi_2/(2-\sec^2\phi_2)$, there is a discontinuity of magnitude σ_2 in s_2, whereas at the corresponding point in the ζ_1 plane, $\zeta_1 = \sec^2\phi_2/(2m^2-\sec^2\phi_2)$, there is no singularity. Applying the theory of residues in each plane, one obtains

$$2mk\sigma_2 = \frac{C+\surd 2E(\sec^2\phi_2-m^2)^{1/2}\cos\phi_2}{mk\tan\phi_2-(\sec^2\phi_2-m^2)^{1/2}}$$

and
$$C-\surd 2mkE\sin\phi_2 = 0,$$

whence
$$E = \surd 2mk\sigma_2\sec\phi_2 \tag{30}$$

and
$$C = 2m^2k^2\sigma_2\tan\phi_2. \tag{31}$$

A similar argument based on the singularity in the ζ_2 plane at $\zeta_2 = m^2\sec^2\phi_1/(2-m^2\sec^2\phi_1)$, corresponding to H, where there is a discontinuity σ_2 in s_2, and on the singularity at the corresponding point K in the ζ_1 plane where s_1 changes by $(\sigma_1-\sigma_1')$, gives

$$B = -2m^4k^2\sigma_1\tan\phi_1 \tag{32}$$

and
$$D = -\surd 2m^4k^2\sigma_1\sec\phi_1. \tag{33}$$

The functions $dw_2/d\zeta_2$, $dw_1/d\zeta_1$ are now determinate, and the solution of the problem is complete. The pressure at any point can be obtained by integration of (24) and (27) in terms of elliptic functions, but since in most cases only the pressure on the face of the wedge will be of interest, it is easier to evaluate the integrals numerically as required.

4. The special case of a temperature front

When the two media concerned are both composed of the same gas, but at different temperatures, some simplification is possible.

The gas laws then give $k = \rho_1/\rho_2 = T_2/T_1$, $c_1^2/c_2^2 = m^2 = T_1/T_2$, and $m^2k = 1$. It will generally be sufficient to use a first approximation, writing the absolute temperature ratio as $T_2/T_1 = 1+\epsilon$ and retaining only linear terms in ϵ. Then $k = 1+\epsilon$, $m^2 = 1-\epsilon$ and $\sec^2\phi_2 = \sec^2\phi_1(1+\epsilon\tan^2\phi_1)$. Equations (25), (26) and (30) to (33) then lead to

$$P_2 = \frac{4M_1\,\delta\epsilon\zeta_2\sin\phi_1}{(\zeta_2\cos 2\phi_1-1)^2} \tag{34}$$

and

$$Q_2 = \frac{\surd 2M_1\,\delta\epsilon\{(1+\zeta_2)+\sin^2\phi_1(\zeta_2\cos 2\phi_1-1)\}}{\cos^2\phi_1(\zeta_2\cos 2\phi_1-1)^2}, \tag{35}$$

and equation (24) gives

$$\frac{\pi}{M\delta\epsilon}\frac{ds}{d\alpha} = \frac{2\alpha\sin\phi}{(\alpha\cos 2\phi+1)^2(1-\alpha)^{1/2}} - \frac{(1-\alpha)^{1/2}}{(2\alpha)^{1/2}\cos^2\phi(1+\alpha\cos 2\phi)^2} + \\ + \frac{\tan^2\phi}{(2\alpha)^{1/2}(1-\alpha)^{1/2}(1+\alpha\cos 2\phi)} \quad (\theta = \tfrac{1}{2}\pi), \tag{36}$$

where
$$\alpha = -\xi = \lambda^2/(2c^2-\lambda^2) \tag{37}$$

and the suffixes have been dropped, since the use of M_1 for M_2 introduces errors only of smaller order than the terms retained. The corresponding result in the other medium is, from (27),

$$\frac{\pi}{M\delta\epsilon}\frac{ds}{d\alpha} = \frac{2\alpha\sin\phi}{(\alpha\cos 2\phi+1)^2(1-\alpha)^{1/2}} + \frac{(1-\alpha)^{1/2}}{(2\alpha)^{1/2}\cos^2\phi(1+\alpha\cos 2\phi)^2} - \\ - \frac{\tan^2\phi}{(2\alpha)^{1/2}(1-\alpha)^{1/2}(1+\alpha\cos 2\phi)} \qquad (\theta = -\tfrac{1}{2}\pi). \quad (38)$$

Integration of (36) gives

$$\frac{s_2-\sigma_2}{M\delta\epsilon} = \frac{\sin\phi\cos 2\phi}{\sin^3 2\phi} - \frac{2\sin\phi(1-\alpha^2)^{1/2}}{\pi\sin^2 2\phi(1+\alpha\cos 2\phi)} - \frac{2\sin\phi\cos 2\phi}{\pi\sin^3 2\phi}\times \\ \times\sin^{-1}\left(\frac{\alpha+\cos 2\phi}{\alpha\cos 2\phi+1}\right) + \frac{1}{4\cos^3\phi} - \frac{\alpha^{1/2}(1-\alpha)^{1/2}}{2^{1/2}\pi\cos^2\phi(1+\alpha\cos 2\phi)} - \\ - \frac{1}{2\pi\cos^3\phi}\tan^{-1}\left\{\cos\phi\left(\frac{2\alpha}{1-\alpha}\right)^{1/2}\right\} - \frac{\sin^2\phi}{4\cos^3\phi} - \\ - \frac{\sin^2\phi}{2\pi\cos^3\phi}\sin^{-1}\left\{\frac{1-\alpha(1+2\cos^2\phi)}{1+\alpha\cos 2\phi}\right\}, \quad (39)$$

and a similar result follows from (37).

5. Application to general symmetric aerofoils

In the linearized theory, superposition of results is allowable, and the motion of a general symmetric aerofoil can be obtained by regarding it as being made up of a number of wedges.

Define a function $G(\lambda)$, $-\infty < \lambda < \infty$, as follows. For $0 < \lambda < \infty$, the value of $\delta G(\lambda)$ is equal to the condensation, s_2, in the above solution on the line $\theta = \frac{1}{2}\pi$ at the point λ. For $-\infty < \lambda < 0$, the value of $\delta G(\lambda)$ is the condensation at $\theta = -\frac{1}{2}\pi$, λ.

Now consider a symmetric aerofoil defined by

$$-X(y) < x < X(y), \qquad -a < y < 0,$$

where x, y are cartesian coordinates moving with the aerofoil and based on its leading edge, and write

$$\psi(y) = -dx/dy$$

for the slope of the surface, assumed everywhere small.

The condensation on the aerofoil before the leading edge breaks the surface of separation of the media is the same as in steady motion in medium 1. When the aerofoil is part way through the interface the condensation due to that part which has passed into the second medium is given by the foregoing theory, but the flow due to the part which has not yet left the medium 1 is still the steady flow appropriate to medium 1. Consider the conditions at a time t, less than a/V, after the leading edge breaks the surface, then the foregoing arguments give, on the aerofoil,

$$s = \psi_0 G\left(\frac{Vt+y}{ct}\right) - \int_{-Vt}^{0}\{\psi(0)-\psi(y^*)\}\times \\ \times G\left(\frac{V^2t+Vy-Vy^*}{Vct+cy^*}\right)dy^* \quad (0 > y > Vt), \quad (40)$$

and

$$s = \psi_0 G\left(\frac{Vt+y}{ct}\right) - \int_{-Vt}^{0} \{\psi(0)-\psi(y^*)\} G\left(\frac{V^2t+Vy-Vy^*}{Vct+cy^*}\right) dy^* -$$
$$- M \sec\phi_1 \int_{y}^{-Vt} \{\psi(0)-\psi(y^*)\}\, dy^* \quad (-Vt > y > -a), \quad (41)$$

where V is, as before, the (supersonic) velocity of the aerofoil. When the aerofoil has completely passed the interface only the expression (40) is required.

6. Numerical results for a double wedge passing through a temperature front

As an example, consider the motion through a temperature front of a double wedge, of angle δ and length $2b$.

λ	H	$\int^{1\cdot 0} H\,d\lambda$	λ	H	$\int^{1\cdot 0} H\,d\lambda$
1·0 to 1·5	−0·1341	0·00000	−0·05	−0·3978	−0·58093
0·95	−0·4896	−0·01559	−0·10	−0·3775	−0·60031
0·90	−0·5867	−0·04250	−0·15	−0·3567	−0·61866
0·85	−0·6328	−0·07299	−0·20	−0·3359	−0·63598
0·80	−0·6557	−0·10520	−0·25	−0·3153	−0·65226
0·75	−0·6654	−0·13823	−0·30	−0·2948	−0·66751
0·70	−0·6636	−0·17145	−0·35	−0·2744	−0·68174
0·65	−0·6570	−0·20447	−0·40	−0·2532	−0·69493
0·60	−0·6461	−0·22704	−0·45	−0·2415	−0·70730
0·55	−0·6322	−0·26900	−0·50	−0·2139	−0·71868
0·50	−0·6167	−0·30022	−0·55	−0·1939	−0·72888
0·45	−0·5960	−0·33054	−0·60	−0·1779	−0·73818
0·40	−0·5822	−0·36000	−0·65	−0·1535	−0·74646
0·35	−0·5623	−0·38861	−0·70	−0·1335	−0·75363
0·30	−0·5424	−0·41623	−0·75	−0·1124	−0·75978
0·25	−0·5223	−0·44284	−0·80	−0·0935	−0·76493
0·20	−0·5017	−0·46844	−0·85	−0·0738	−0·76911
0·15	−0·4810	−0·49301	−0·90	−0·0539	−0·77230
0·10	−0·4602	−0·51654	−0·95	−0·0334	−0·77449
0·05	−0·4399	−0·53904	−1·00 to −∞	−0·0000	−0·77532
0·00	−0·4188	−0·56051			

Table 1.

It is convenient to tabulate a function $H(\lambda)$ defined by

$$H(\lambda) = \frac{G(\lambda)-\sigma_1}{\epsilon\sigma_1}.$$

Then (39) leads to the results in table 1.

The drag on the aerofoil then follows from (40) and (41). In table 2 a correction factor e is given such that the total drag is

$$\rho c_1^2 \sigma_1 b\delta\{2-e\epsilon\}. \quad (42)$$

The correction factor is tabulated against Vt/b, that is the number of half lengths by which the leading edge has passed the interface.

Vt/b	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	3	4	5	6
e	0·2808	0·2752	0·3103	0·5476	0·0817	−0·0209	0·1712	0·2682

Table 2.

When Vt/b exceeds 6, the transient effects no longer influence the drag and the drag is constant and equal to that for steady motion in the new medium.

References

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